# Tutorial 8

We recall some definitions.

**Definition 1.** Let $A$ be a subset of the state space $S$. The *hitting time* $T_A$ of $A$ is defined by

$$T_A := \min\{n \geq 1 : X_n \in A\}.$$

For a singleton $A = \{a\}$, we will denote the hitting time by $T_a$ rather than the cumbersome notation $T_{\{a\}}$.

**Notation 1.** Set

$$\rho_{xy} := \mathbb{P}_x(T_y < \infty)$$

i.e. $\rho_{xy}$ denotes the probability that a Markov chain starting at $x$ will be in some other state $y$ at some positive time. In particular $\rho_{yy}$ denotes the probability that a Markov chain starting at $y$ will ever return to $y$.

The notation $\mathbb{P}_x$ denotes probabilities of various events defined in terms of a Markov chain starting at $x$ e.g.

$$\begin{aligned}
&\mathbb{P}(X_{n+1} \in B_1, \ldots, X_{n+m} \in B_m \mid X_0 \in A_0, \ldots, X_{n-1} \in A_{n-1}, X_n = x) \\
&= \mathbb{P}(X_{n+1} \in B_1, \ldots, X_{n+m} \in B_m \mid X_n = x) \\
&= \mathbb{P}_x(X_1 \in B_1, \ldots, X_m \in B_m)
\end{aligned}$$

**Definition 2.** A state $y$ is called *recurrent* if $\rho_{yy} = 1$ and *transient* if $\rho_{yy} < 1$.

We say a state $x$ *leads to* another state $y$ if $\rho_{xy} > 0$.

**Definition 3.** A non-empty subset $C \subseteq S$ is said to be *closed* if no state inside of $C$ leads to any state outside of $C$.

A closed set $C$ is called *irreducible* if $x$ leads to $y$ for all $x, y \in C$.

A Markov chain is called *irreducible* if its state space is irreducible i.e. every state leads back to itself and also to every other state.

Hence an irreducible Markov chain is necessarily either recurrent or transient.

**Recurrence of Queuing Chains**

Let us first review about queuing chains; you may also see example 5 in section 1.3 of the textbook.

Consider your favourite restaurant at CUHK. During lunch time, lots of students would come in and form a waiting line.

Suppose the crowed has the following pattern: when you walk in and you see there are some students waiting in line, then exactly one student (i.e. the one in the front most) will be served during your observation; however, if you saw there was no waiting line, then no one will be served (of course! Unless you step in and order something.).

Let $\xi_n$ denote the number of students arriving at time $n$ and we assume that $\xi_1, \xi_2, \ldots$ are independent non-negative integer-valued random variables having a common density $f$ and hence a common mean $\mu$. This assumption of independence is natural: number of students that come in at time 1 and at time 2 are expected to be independent, because we can plausibly assume most of time these two groups of people do not know each other, for example; plus there are tons of factors that would affect the number of students coming to a restaurant e.g. the time classes end and the time buses arrive etc.; these effects should be assumed to be negligibly small otherwise we would not be able to analyze such complicated situations. In fact, analyzing such an ideal model can provide quite a lot useful information.

Let $X_n$ denote the number of students present at time $n$. Then

$$X_{n+1} = \begin{cases} \xi_{n+1}, & \text{if } X_n = 0 \\ X_n + \xi_{n+1} - 1, & \text{if } X_n \geq 1 \end{cases}$$

where the $-1$ indicates the exact one student that was being served at time $n$.

$X_n$ is a Markov chain with transition function

$$P(x,y) = \mathbb{P}(X_{n+1} = y \mid X_n = x) = \begin{cases} f(y), & x = 0 \\ f(y-x+1), & x \geq 1 \end{cases}$$

Now, suppose that this queuing chain is irreducible. Then by exercise 37 from chapter 1 of the textbook [1], we have that $f(0) > 0$ and $f(0) + f(1) < 1$. Also, by section 1.8.2 and section 1.9.2, we know that the chain is recurrent if $\mu \leq 1$ and transient if $\mu > 1$.

**Claim.** We will show that in the recurrent case

$$m_0 = \frac{1}{1-\mu} \tag{1}$$

where $\mu := \mathbb{E}_0[T_0]$.

*Remark.* It would then follow from (1) that if $\mu < 1$, then $m_0 < \infty$ and hence 0 is a positive recurrent state. Thus by irreducibility the chain is positive recurrent.

On the other hand, if $\mu = 1$, then $m_0 = \infty$ and hence 0 is a null recurrent state. Thus we conclude that the chain is null recurrent in this case.

Therefore, an irreducible queuing chain is positive recurrent if $\mu < 1$ and null recurrent if $\mu = 1$, and transient if $\mu > 1$.

Now let us prove the claim.

*Proof.* Consider the chain starting at some positive integer $x > 0$. Then $T_{x-1}$ denotes the time to go from state $x$ to state $x-1$, and $T_{y-1} - T_y, 1 \leq y \leq x-1$ denotes the time to go from state $y$ to state $y-1$.

Since the chain goes at most one step to the left at a time (by definition!), the Markov property insures that the random variables

$$T_{x-1}, T_{x-2} - T_{x-1}, \ldots, T_0 - T_1$$

are independent. They are also identically distributed; for each of them is distributed as [2]

$$\min\{n \geq 1 : \xi_1 + \cdots + \xi_n = n-1\}.$$

The probability generation function $G(t)$ of the time to go from state 1 to state 0 is

$$G(t) = \sum_{n=1}^{\infty} t^n \mathbb{P}_1(T_0 = n)$$

Now, recall that the probability generating function of a sum of independent non-negative integer-valued random variables is the product of their respective probability generating functions.

If the chain starts at $x$, then

$$T_0 = T_{x-1} + (T_{x-2} - T_{x-1}) + \cdots + (T_0 - T_1)$$

[1] See also solutions to homework 3.

[2] To understand this, you may substitute a concrete example: say the chain hits $x-1$ at $m$ and then hits $x-2$ at some later time $m+5$. In this case $T_{x-2} - T_{x-1} = 5$ and in order for this to happen, it must be true that $\xi_{m+1} + \cdots \xi_{m+5} = 5 - 1$ (do a model computation using the definition of $X_n$!).

is the sum of $x$ independent random variables, each having the probability generating function $G(t)$. Thus it follows that

$$[G(t)]^x = \sum_{n=1}^{\infty} t^n \mathbb{P}_x(T_0 = n). \tag{2}$$

**Step 1.** We will now show that

$$G(t) = t\Phi(G(t))$$

where $\Phi$ is the probability generating function of $f$.

By shifting the index, we can rewrite

$$G(t) = \sum_{n=0}^{\infty} t^{n+1}\mathbb{P}_1(T_0 = n+1) = tP(1,0) + t\sum_{n=1}^{\infty} t^n \mathbb{P}_1(T_0 = n+1).$$

By recalling the property [3]

$$\mathbb{P}_x(T_y = n+1) = \sum_{z \neq y} P(x,z)\mathbb{P}_z(T_y = n) \tag{3}$$

we have

$$\begin{aligned}
G(t) &= tP(1,0) + t\sum_{n=1}^{\infty} t^n \mathbb{P}_1(T_0 = n+1) \\
&= tP(1,0) + t\sum_{n=1}^{\infty} t^n \sum_{z\neq 0} P(1,z)\mathbb{P}_z(T_0 = n) \quad \text{take } x=1, y=0 \text{ in (3)} \\
&= tP(1,0) + t\sum_{z\neq 0} P(1,z) \sum_{n=1}^{\infty} t^n \mathbb{P}_z(T_0 = n) \quad \text{(interchange the order)} \\
&= tP(1,0) + t\sum_{z\neq 0} P(1,z)[G(t)]^z \quad \text{by (2)} \\
&= t\left[f(0) + \sum_{z\neq 0} f(z)[G(t)]^z\right] \quad \text{since } P(1,z) = f(z) \text{ for } z \geq 0 \\
&= t\Phi[G(t)]
\end{aligned}$$

**Step 2.** Next, we will show that

$$\lim_{t\to 1} G'(t) = \frac{1}{1-\mu}.$$

Indeed, by differentiating with respect to $t$, we get

$$G'(t) = \Phi[G(t)] + tG'(t)\Phi'[G(t)].$$

Solving for $G'(t)$, we get

$$G'(t) = \frac{\Phi[G(t)]}{1 - t\Phi'[G(t)]}.$$

Since the probability generating functions go to 1 as $t \to 1$ and

$$\lim_{t\to 1}\Phi'(t) = \lim_{t\to 1}\sum_{x=1}^{\infty} x f(x) t^{x-1} = \sum_{x=1}^{\infty} x f(x) = \mu,$$

[3] See (29) in chapter 1 of the textbook.

the result follows.

**Step 3.** Finally, we will show

$$\lim_{t \to 1} G'(t) = m_0$$

and the proof will be complete.

Since $P(1,x) = P(0,x)$ for $x \geq 0$, it follows from (3) that

$$\mathbb{P}_1(T_0 = n+1) = \sum_{z \geq 1} P(1,z)\mathbb{P}_z(T_0 = n) = \sum_{z \neq 1} P(0,z)\mathbb{P}_z(T_0 = n) = \mathbb{P}_0(T_0 = n+1).$$

Hence,

$$G(t) = \sum_{n=1}^{\infty} t^n \mathbb{P}_1(T_0 = n) = \sum_{n=1}^{\infty} t^n \mathbb{P}_0(T_0 = n).$$

Thus, we have

$$\begin{aligned}
\lim_{t \to 1} G'(t) &= \lim_{t \to 1} \sum_{n=1}^{\infty} n t^{n-1} \mathbb{P}_0(T_0 = n) \\
&= \sum_{n=1}^{\infty} n \mathbb{P}_0(T_0 = n) \\
&= \mathbb{E}_0[T_0] \\
&= m_0
\end{aligned}$$

□